

k -core organization of complex networks

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We analytically describe the architecture of randomly damaged uncorrelated networks as a set of successively enclosed substructures — k -cores. The k -core is the largest subgraph where vertices have at least k interconnections. We find the structure of k -cores, their sizes, and their birth points — the bootstrap percolation thresholds. We show that in networks with a finite mean number z_2 of the second-nearest neighbors, the emergence of a k -core is a hybrid phase transition. In contrast, if z_2 diverges, the networks contain an infinite sequence of k -cores which are ultra-robust against random damage.

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Introduction.—Extracting and indexing highly interconnected parts of complex networks—communities, cliques, cores, etc.—as well as finding relations between these substructures is an issue of topical interest in network research, see, e.g., Refs. [1, 2]. This decomposition helps one to describe the complex topologies of real-world networks. In this respect, the notion of k -core is of fundamental importance [3, 4]. The k -core may be obtained in the following way. Remove from a graph all vertices of degree less than k . Some of the rest vertices may remain with less than k edges. Then remove these vertices, and so on until no further removal is possible. The result, if it exists, is the k -core. Thus, a network is organized as a set of successively enclosed k -cores, similarly to a Russian nesting doll.

The k -core decomposition was recently applied to a number of real-world networks (the Internet, the WWW, cellular networks, etc.) [5, 6, 7] and was turned out to be an important tool for visualization of complex networks and interpretation of cooperative processes in them. Rich k -core architectures of real networks was revealed. Furthermore, a k -core related Jellyfish model [8] is one of the popular models of the Autonomous System graph of the Internet. The notion of the k -core is a natural generalization of the giant connected component in the ordinary percolation [9, 10, 11] (for another possible generalization, see clique percolation in Ref. [12]). Impressively, the giant connected component of an infinite network with a heavy-tailed degree distribution is robust against random damage of the net. The k -core percolation implies the emergence of a giant k -core below a threshold concentration of vertices or edges removed at random. In physics, the k -core percolation (bootstrap percolation) on the Bethe lattice was introduced in Ref. [13] for describing some magnetic materials. Note that the $k \geq 3$ -core percolation is an unusual, hybrid phase transition with a jump of the order parameter as at a first order phase transition but also with strong critical fluctuations as at a continuous phase transition [13, 14]. The k -core decomposition of a random graph was formulated as a mathematical problem in Refs. [3, 4]. This attracted

much attention of mathematicians [15, 16], but actually only the criteria of emergence of k -cores in basic random networks were found.

In this Letter we derive exact equations describing the k -core organization of a randomly damaged uncorrelated network with an arbitrary degree distribution. This allows us to obtain the sizes and other structural characteristics of k -cores in a variety of damaged and undamaged random networks and find the nature of the k -core percolation in complex networks. We apply our general results to the classical random graphs and to scale-free networks, in particular, to empirical router-level Internet maps. We find that not only the giant connected components in infinite networks with slowly decreasing degree distributions are resilient against random damage, as was known, but their entire k -core architectures are robust.

Basic equations.—We consider an uncorrelated network—a maximally random graph with a given degree distribution $P(q)$ —the so-called configuration model. We assume that a fraction $Q \equiv 1 - p$ of the vertices in this network are removed at random. The k -core extracting procedure results in the structure of the network with a k -core depicted in Fig. 1.

Taking into account the tree-like structure of the infinite sparse configuration model shows that the k -core coincides with the infinite $(k-1)$ -ary subtree [17]. (The m -ary tree is a tree, where all vertices have branching at

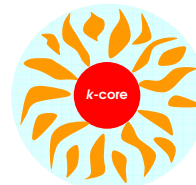


FIG. 1: The structure of a network with a k -core. The k -core is the internal circle. Vertices of degrees smaller than k form the light-grey area. The dark-grey regions are numerous finite clusters with those vertices of degrees $q \geq k$, which do not belong to the k -core. These clusters are either connected to the k -core by less than k edges or isolated from it.

least m .) Let R be the probability that a given end of an edge of a network is not the root of an infinite $(k-1)$ -ary subtree. Then a vertex belongs to the k -core if at least k its neighbors are roots of infinite $(k-1)$ -ary subtrees. So the probability that a vertex is in the k -core is

$$M(k) = p \sum_{q \geq k} P(q) \sum_{n=k}^q C_n^q R^{q-n} (1-R)^n, \quad (1)$$

where $C_n^m = m!/(m-n)!n!$. Note that for the ordinary percolation we must set $k=1$ in this equation.

An end of an edge is not a root of an infinite $(k-1)$ -ary subtree if at most $k-2$ its children branches are roots of infinite $(k-1)$ -ary subtrees. This leads to the following equation for R :

$$R = 1 - p + p \sum_{n=0}^{k-2} \left[\sum_{i=n}^{\infty} \frac{(i+1)P(i+1)}{z_1} C_n^i R^{i-n} (1-R)^n \right]. \quad (2)$$

Let us explain this equation. (i) The first term, $1-p \equiv Q$, is the probability that the end of the edge is unoccupied. (ii) $C_n^i R^{i-n} (1-R)^n$ is the probability that if a given end of the edge has i children (i.e., other edges than the starting edge), then exactly n of them are roots of infinite $(k-1)$ -ary subtrees. $(i+1)P(i+1)/z_1$ is the probability that a randomly chosen edge leads to a vertex with branching i . $z_1 = \sum_q qP(q)$ is the mean number of the nearest neighbors of a vertex in the graph. Thus, in the square brackets, we present the probability that a given end of the edge has exactly n edges, which are roots of infinite $(k-1)$ -ary subtrees. (iii) Finally, we take into account that n must be at most $k-2$.

The sum $\sum_{n=0}^{k-2}$ in Eq. (2) may be rewritten as:

$$\Phi_k(R) = \sum_{n=0}^{k-2} \frac{(1-R)^n}{n!} \frac{d^n}{dR^n} G_1(R), \quad (3)$$

where $G_1(x) = z_1^{-1} \sum_q P(q) q x^{q-1} = z_1^{-1} dG_0(x)/dx$, and $G_0(x) = \sum_q P(q) x^q$ [18]. Then Eq. (2) takes the form:

$$R = 1 - p + p \Phi_k(R). \quad (4)$$

In the case $p=1$, Eq. (4) was recently obtained in [16]. If Eq. (4) has only the trivial solution $R=1$, there is no giant k -core. The emergence of a nontrivial solution corresponds to the birth of the giant k -core. It is the lowest nontrivial solution $R < 1$ that describes the k -core.

Let us define a function

$$f_k(R) = [1 - \Phi_k(R)] / (1 - R). \quad (5)$$

This function is positive in the range $R \in [0, 1)$ and, in networks with a finite mean number of the second neighbors of a vertex, $z_2 = \sum_q q(q-1)P(q)$, it tends to zero in the limit $R \rightarrow 1$ as $f_k(R) \propto (1-R)^{k-2}$. In terms of the function $f_k(R)$, Eq. (2) is especially simple:

$$p f_k(R) = 1. \quad (6)$$

Depending on $P(q)$, with increasing R , $f_k(R)$ either (i) monotonously decreases from $f_k(0) < 1$ to $f_k(1) = 0$, or (ii) at first increases, then approaches a maximum at $R_{\max} \in (0, 1)$, and finally tends to zero at $R \rightarrow 1$. Therefore Eq. (6) has a non-trivial solution $R < 1$ if

$$p \max_{R \in [0, 1)} f_k(R) \geq 1. \quad (7)$$

This is the criterion for the emergence of the giant k -core in a randomly damaged uncorrelated network. The equality in Eq. (7) takes place at a critical concentration $p_c(k)$ when the line $y(R) = 1/p_c(k)$ touches the maximum of $f_k(R)$. Therefore the threshold of the k -core percolation is determined by two equations:

$$p_c(k) = 1/f_k(R_{\max}), \quad 0 = f'_k(R_{\max}). \quad (8)$$

R_{\max} is the value of the order parameter at the birth point of the k -core. At $p < p_c(k)$ there is only the trivial solution $R=1$.

At $k=2$, Eq. (4) describes the ordinary percolation in a random uncorrelated graph [10, 11]. In this case, in infinite networks we have $R_{\max} \rightarrow 1$, and the criterion (7) is reduced to the standard condition for existence of the giant connected component: $pG'_1(1) = pz_2/z_1 \geq 1$.

Let us find R near the $k \geq 3$ -core percolation transition in a network with a finite z_2 . We examine Eq. (4) for $R = R_{\max} + r$ and $p = p_c(k) + \epsilon$ with $|\epsilon, r| \ll 1$. Note that at $k \geq 3$, $\Phi_k(R)$ is an analytical function in the range $R \in [0, 1)$. It means that the expansion of $\Phi_k(R+r)$ over r contains no singular term at $R \in [0, 1)$. Substituting this expansion into Eq. (4), in the leading order, we find

$$R_{\max} - R \propto [p - p_c(k)]^{1/2}, \quad (9)$$

i.e., the combination of a jump and the square root critical singularity. The origin of this singularity is an intriguing problem of the hybrid phase transition.

The structure of the k -core is essentially determined by its degree distribution which we find to be

$$P_k(q) = \frac{p}{M(k)} \sum_{q' \geq q} P(q') C_q^{q'} R^{q'-q} (1-R)^q. \quad (10)$$

The mean degree of the k -core vertices is $z_1(k) = \sum_{q \geq k} P_k(q) q$. The k -core of a given graph contains the $k+1$ -core as a subgraph. Vertices which belong to the k -core, but do not belong to the $k+1$ -core, form the k -shell of the relative size $S(k) = M(k) - M(k+1)$.

We apply our general results to two basic networks.

Erdős-Rényi (ER) graphs.—These random graphs have the Poisson degree distribution $P(q) = z_1^q \exp(-z_1)/q!$, where z_1 is the mean degree. In this case, $G_0(x) = G_1(x) = \exp[z_1(x-1)]$. In Eq. (4), $\Phi_k(R) = \Gamma[k-1, z_1(1-R)]/\Gamma(k-1)$, where

$\Gamma(n, x)$ is the incomplete gamma function. From Eq. (1) we get the size of the k -core:

$$M(k) = p\{1 - \Gamma[k, z_1(1 - R)]/\Gamma(k)\}, \quad (11)$$

where R is the solution of Eq. (4). The degree distribution in the k -core is $P_k(q \geq k) = pz_1^q(1-R)^q e^{-z_1(1-R)}/[M(k)q!]$. Our numerical calculations revealed that at $p = 1$, the highest k -core increases almost linearly with z_1 , namely, $k_h \approx 0.78z_1$ at $z_1 \lesssim 500$. Furthermore, the mean degree $z_1(k)$ in the k -core weakly depends on k : $z_1(k) \approx z_1$.

Fig. 2 shows the dependence of the size of the k -cores, $M(k)$, on the concentration $Q = 1 - p$ of the vertices removed at random. Note that counterintuitively, it is the highest k -core—the central, most interconnected part of a network—that is destroyed primarily. The inset of Fig. 2 shows that with increasing damage Q , the mean degree $z_1(k)$ decreases. The k -cores disappear consecutively, starting from the highest core. The k -core structure of the undamaged ER graphs is displayed in Fig. 3.

Scale-free networks.—We consider uncorrelated networks with a degree distribution $P(q) \propto (q + c)^{-\gamma}$. Let us start with the case of $\gamma > 3$, where z_2 is finite. It turns out that the existence of k -cores is determined by the complete form of the degree distribution including its low degree region. It was proved in Ref. [16] that there is no $k \geq 3$ -core in a graph with the minimal degree $q_0 = 1$, $\gamma \geq 3$, and $c = 0$. We find that the k -cores emerge as c increases. The k -core structure of scale-free graphs is represented in Fig. 3. The relative sizes of the giant k -cores in the scale-free networks are smaller than in the ER graphs. As z_2 is finite, the $k \geq 3$ -core percolation at $\gamma > 3$ is the hybrid phase transition. This is in contrast to the ordinary percolation in scale-free networks, where behavior is non-standard if $\gamma \leq 4$ [10].

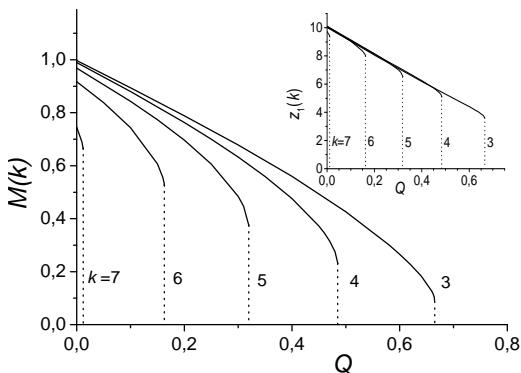


FIG. 2: The size of the k -core, $M(k)$, in the Erdős-Rényi random graph with the mean degree $z_1 = 10$ versus the concentration Q of vertices removed at random. The highest core disappears at a very low concentration $Q \approx 1.2\%$ in contrast to the ordinary percolation threshold $Q \approx 90\%$. The inset shows the mean degree $z_1(k)$ of vertices in the k -core.

The case $2 < \gamma \leq 3$ is realized in most important real-world networks. With γ in this range, z_2 diverges if $N \rightarrow \infty$. In the leading order in $1 - R \ll 1$, Eq. (5) gives $f_k(R) \cong (q_0/k)^{\gamma-2}(1-R)^{-(3-\gamma)}$. From Eq. (6) we find the order parameter R . Substituting this solution into Eq. (1), in the leading order in $1 - R$ we find that the size of the k -core decreases with increasing k :

$$M(k) = p[q_0(1-R)/k]^{\gamma-1} = p^{2/(3-\gamma)}(q_0/k)^{(\gamma-1)/(3-\gamma)}. \quad (12)$$

The divergence of $f_k(R)$ at $R \rightarrow 1$ means that the percolation threshold $p_c(k)$ tends to zero as $N \rightarrow \infty$. The k -core percolation transition in this limit is of infinite order similarly to the ordinary percolation [10]. As $k_h(N \rightarrow \infty) \rightarrow \infty$, there is an infinite sequence of successively enclosed k -cores. One has to remove at random almost all vertices in order to destroy any of these cores.

Eq. (10) allows us to find the degree distribution of k -cores in scale-free networks. For $\gamma > 2$ and $k \gg 1$, $P_k(q \gg k) \approx (\gamma - 1)k^{\gamma-1}q^{-\gamma}$. The mean degree $z_1(k)$ in the k -core grows linearly with k : $z_1(k) \approx kz_1/q_0$ in contrast to the Erdős-Rényi graphs.

Finite-size effect.—The finiteness of the scale-free networks with $2 < \gamma < 3$ essentially determines their k -core organization. We introduce a size dependent cutoff $q_{\text{cut}}(N)$ of the degree distribution. Here $q_{\text{cut}}(N)$ depends on details of a specific network. For example, for the configuration model without multiple connections, the dependence $q_{\text{cut}}(N) \sim \sqrt{N}$ is usually used if $2 < \gamma < 3$. It is this function that must be substituted into Eqs. (13), (14), and (15) below. A detailed analysis of Eq. (5) shows that the cutoff dramatically changes the behavior

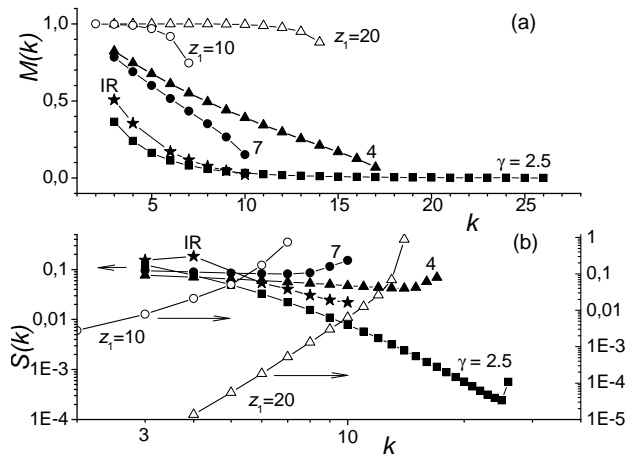


FIG. 3: The relative sizes of the k -cores, $M(k)$, panel (a), and k -shells, $S(k)$, panel (b), in the Erdős-Rényi graphs with $z_1 = 10$ and 20 ; scale-free networks with $\gamma = 2.5, 4$ and 7 , and an uncorrelated network with the degree distribution of the router-level Internet map (IR). The minimum degree in the scale-free networks is $q_0 = 1$. In the case $\gamma = 2.5$, the maximum degree in the network is $q_{\text{cut}} = 2000$, and $c = 2$; for $\gamma = 4$ and 7 , $c = 30$ and 50 , respectively.

of the function $f_k(R)$ near $R = 1$. $f_k(R)$ has a maximum at $R_{\max} \cong 1 - (3 - \gamma)^{-1/(\gamma-2)} k/q_{\text{cut}}$ and tends to zero at $R \rightarrow 1$ instead of divergence. As a result, the k -core percolation again becomes to be the hybrid phase transition. The cutoff determines the highest k -core:

$$k_h \cong p(\gamma - 2)(3 - \gamma)^{(3-\gamma)/(\gamma-2)} q_{\text{cut}}(q_0/q_{\text{cut}})^{\gamma-2}. \quad (13)$$

The sizes of the k -core at $q_0 \ll k \ll k_h$ are given by Eq. (12). The relative size of the highest k -core is

$$M(k_h) \cong p[(3 - \gamma)^{-(\gamma-1)/(\gamma-2)} - 1](q_0/q_{\text{cut}})^{\gamma-1}. \quad (14)$$

Finally, the threshold of the k -core percolation is

$$p_c(k) = 1/f_k(R_{\max}) \cong k/k_h. \quad (15)$$

If $k \rightarrow k_h$, then $p_c(k) \rightarrow 1$, i.e. even minor random damage destroys the highest k_h -core. By using exact Eqs. (2) and (1), we calculated numerically $M(k)$ and $S(k)$ for a scale-free network with $\gamma = 2.5$, see Fig. 3. These curves agree with asymptotic expressions (12) and (14).

k-core organization of the router-level Internet.—We consider the router-level Internet which has lower degree-degree correlations than the Internet at the Autonomous Systems (AS) level. We substitute the empirical degree distribution of the router-level Internet as seen in skitter and iffinder measurements [19] into our exact equations and compare our results with the direct k -core decomposition of this network. The calculated sizes of k -cores and k -shells are shown in Fig. 3. The calculated dependence $S(k)$ [Fig. 3(b), the IR curve] is surprisingly similar to the dependence obtained by the direct k -core decomposition of, actually, a different network—the AS-level Internet—in Ref. [6]. On the other hand, one can see in Fig. 3 that the highest k -core with $k_h = 10$ occupies about 2% of the network, while a direct k -core decomposition of the same router-level Internet map in Ref. [5] revealed k -cores up to $k_h = 32$. This difference indicates the significance of degree-degree correlations, which we neglected.

Discussion and conclusions.—It is important to indicate a quantity critically divergent at the k -core's birth point. This is a mean size of a cluster of vertices of the k -core with exactly k connections inside of the k -core. One may show that it diverges as $-dM(k)/dp \sim (p - p_c)^{-1/2}$ and that the size distribution of these clusters is a power law at the critical point.

One should note that the k -core (or bootstrap) percolation is not related to the recently introduced k -clique percolation [12] despite of the seemingly similar terms. The k -clique percolation is due to the overlapping of k -cliques—full subgraphs of k vertices—by $k - 1$ vertices. Therefore, the k -clique percolation is impossible in sparse networks with few loops, e.g., in the configuration model and in classical random graphs, considered here.

In summary, we have developed the theory of k -core percolation in damaged uncorrelated networks. We have

found that if the second moment of the degree distribution of a network is finite, the k -core transition has the hybrid nature. In contrast, in the networks with infinite z_2 , instead of the hybrid transition, we have observed an infinite order transition, similarly to the ordinary percolation in this situation. All k -cores in these networks are extremely robust against random damage. It indicates the remarkable robustness of the entire k -core architectures of infinite networks with $\gamma \leq 3$. Nonetheless, we have observed that the finite networks are less robust, and increasing failures successively destroy k -cores starting from the highest one. Our results can be applied to numerous cooperative models on networks: a formation of highly connected communities in social networks, the spread of diseases, and many others.

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